Almost Gorenstein Rees algebras

based on the works jointly with

Shiro Goto, Naoyuki Matsuoka, and Ken-ichi Yoshida

Naoki Endo (Waseda University)

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Introduction

What is the Rees algebra?

For a commutative ring $R$ and an ideal $I$ in $R$, set

$$\mathcal{R}(I) = R[lt] = \sum_{n \geq 0} l^n t^n \subseteq R[t]$$

$$G(I) = \mathcal{R}(I)/I\mathcal{R}(I) = \bigoplus_{n \geq 0} l^n/I^{n+1}. $$
Example 1.1

Let \( R = k[X_1, X_2, \ldots, X_d] \) (\( d \geq 1 \)) and \( I = (X_1, X_2, \ldots, X_d) \). Then

\[
\mathcal{R}(I) \cong k[X_1, X_2, \ldots, X_d, Y_1, Y_2, \ldots, Y_d]/I_2 (X_1 X_2 \cdots X_d)
\]

More generally, if

- \((R, m)\) a CM local ring
- \(Q = (a_1, a_2, \ldots, a_d)\) a parameter ideal in \( R \)

then

\[
\mathcal{R}(Q) \cong R[Y_1, Y_2, \ldots, Y_d]/I_2 (a_1 a_2 \cdots a_d)
\]

is a CM ring, where \( d = \dim R \).
Preceding results

**Theorem 1.2 (Goto-Shimoda)**

Let \((R, \mathfrak{m})\) be a CM local ring with \(d = \dim R \geq 1, \sqrt{I} = \mathfrak{m}\). Then

\[ \mathcal{R}(I) \text{ is a CM ring} \iff G(I) \text{ is a CM ring}, \quad a(G(I)) < 0 \]

where

\[ a(G(I)) = \sup \{ n \in \mathbb{Z} \mid [H^d_M(G(I))]_n \neq (0) \}, \quad M = \mathfrak{m}\mathcal{R}(I) + \mathcal{R}(I)_+. \]

**Example 1.3**

Let \((R, \mathfrak{m})\) be a RLR with \(\dim R = 2\), \(I\) an ideal of \(R\) s.t. \(I = \overline{I}\) and \(\sqrt{I} = \mathfrak{m}\). Then \(\mathcal{R}(I)\) is a CM ring.
**Theorem 1.4 (Goto-Nishida, Goto-Shimoda, Ikeda)**

Let \((R, \mathfrak{m})\) be a CM local ring with \(d = \dim R \geq 2, \sqrt{I} = \mathfrak{m}\). Then

\[ R(I) \text{ is Gorenstein } \iff G(I) \text{ is Gorenstein, } a(G(I)) = -2. \]

When this is the case, \(R\) is a Gorenstein ring.

Thus, if \(R\) is a CM local ring with \(\dim R \geq 2\), \(Q\) is a parameter ideal, then

\[ R(Q) \text{ is Gorenstein } \iff R \text{ is Gorenstein, } \dim R = 2. \]

Moreover, if \((R, \mathfrak{m})\) is a RLR with \(\dim R = 2\) and \(I = \mathfrak{m}^\ell (\ell \geq 1)\), then

\[ R(I) \text{ is Gorenstein } \iff I = \mathfrak{m}. \]
Question 1.5

When is the Rees algebra $R(I)$ almost Gorenstein?

- $I$ is the ideal generated by a (sub) system of parameters
- $I = \overline{I}$ in a two-dimensional RLR
What is an almost Gorenstein ring?

- In 1997, Barucci and Fröberg defined the notion of almost Gorenstein rings for one-dimensional analytically unramified local rings.
- In 2013, Goto, Matsuoka, and Phuong generalized the notion to arbitrary one-dimensional CM local rings.
- In 2015, Goto, Takahashi, and Taniguchi gave the notion of almost Gorenstein local/graded rings of arbitrary dimension.


**Survey on AG rings**

- $(R, \mathfrak{m})$ a CM local ring with $d = \dim R$, $|R/\mathfrak{m}| = \infty$
- $\exists K_R$ the canonical module of $R$.

**Definition 2.1**

We say that $R$ is an *almost Gorenstein local ring* (abbr. AGL ring), if

$\exists$ an exact sequence

$$0 \to R \to K_R \to C \to 0$$

of $R$-modules s.t. $\mu_R(C) = e(C)$

where

$$e(C) = \lim_{n \to \infty} (d - 1)! \cdot \frac{\ell_R(C/\mathfrak{m}^{n+1}C)}{n^{d-1}}.$$
If $C \neq (0)$, then $C$ is CM and $\dim_R C = d - 1$. Besides

$$\mu_R(C) = e(C) \iff mC = (f_2, f_3, \ldots, f_d)C$$

for $\exists f_2, f_3, \ldots, f_d \in m$. Hence $C$ is an Ulrich $R$-module.

**Example 2.2**

- $k[[t^3, t^4, t^5]].$
- $k[[X, Y, Z]]/(X, Y) \cap (Y, Z) \cap (Z, X).$
- $k[[t^3, t^4, t^5]] \ltimes (t^3, t^4, t^5).$
- $k[[t^3, t^4, t^5]] \times_k k[[t^3, t^4, t^5]].$
- 1-dimensional finite CM–representation type.
- 2-dimensional rational singularity.
$R = \bigoplus_{n \geq 0} R_n$ a CM graded ring, $d = \dim R$, $\exists K_R$

$(R_0, m)$ a local ring, $|R_0/m| = \infty$

**Definition 2.3**

We say that $R$ is an **almost Gorenstein graded ring** (abbr. AGG ring), if

$$\exists \quad 0 \to R \to K_R(-a) \to C \to 0$$

of graded $R$-modules s.t. $\mu_R(C) = e(C)$

where $a = a(R)$, $M = mR + R_+$.

- $R$ is an **AGG** ring $\implies R_M$ is an **AGL** ring.
- The converse is **not true** in general.
Example 2.4

Let $S = k[X_{ij} | 1 \leq i \leq m, 1 \leq j \leq n] \ (2 \leq m \leq n)$ and set

$$R = S/I_t(X)$$

where $2 \leq t \leq m$, $X = [X_{ij}]$. Then

$R$ is an AGG ring $\iff m = n$, or $m \neq n$ and $t = m = 2$.

Example 2.5

Let $R = k[X_1, X_2, \ldots, X_d] \ (d \geq 1)$ and $1 \leq n \in \mathbb{Z}$. Then

- If $d \leq 2$, then $R^{(n)} = k[R_n]$ is an AGG ring.
- If $d \geq 3$, then

  $R^{(n)}$ is an AGG ring $\iff n \mid d$, or $d = 3$ and $n = 2$. 
Main results (parameter ideals)

Let

1. \((R, m)\) a CM local ring with \(d = \dim R \geq 3\)
2. \(a_1, a_2, \ldots, a_r \in m\) a subsystem of parameters in \(R\) \((r \geq 3)\)
3. \(Q = (a_1, a_2, \ldots, a_r)\)
4. \(\mathcal{R} = \mathcal{R}(Q) = R[Qt] \subseteq R[t], M = m\mathcal{R} + \mathcal{R}_+\)

Then

1. \(\mathcal{R} \cong R[X_1, X_2, \ldots, X_r]/I_2 \left( \frac{X_1}{a_1} \frac{X_2}{a_2} \cdots \frac{X_r}{a_r} \right)\) is a CM ring.
2. \(\dim \mathcal{R} = d + 1\) and \(a(\mathcal{R}) = -1\).
Theorem 3.1

- \( R \) is an AGG ring \( \iff \) \( R \) is a RLR and \( a_1, a_2, \ldots, a_r \) is a regular system of parameters in \( R \)

- \( R_M \) is an AGL ring \( \iff \) \( R \) is a RLR

Key for the proof

- The Eagon-Northcott complex
- Proposition 3.2
Proposition 3.2

Let $(B, \mathfrak{n})$ be a Gorenstein local ring, $I$ an ideal of $B$. Suppose that $A = B/I$ is a non-Gorenstein AGL ring. If $pd_B A < \infty$, then $B$ is a RLR.

Proof. May assume $|B/\mathfrak{n}| = \infty$. Choose an exact sequence

$$0 \to A \to K_A \to C \to 0$$

s.t. $C$ is an Ulrich $A$-module. Then $pd_B C < \infty$. Take an $A$-regular sequence $f_1, f_2, \ldots, f_{d-1} \in \mathfrak{n}$ s.t.

$$\mathfrak{n}C = (f_1, f_2, \ldots, f_{d-1})C$$

where $d = \dim A$. Set $q = (f_1, f_2, \ldots, f_{d-1})$. Since $f_1, f_2, \ldots, f_{d-1}$ is a regular sequence on $C$, $pd_B C/qC < \infty$. Hence $B$ is a RLR, because $C/qC = C/\mathfrak{n}C$ is a vector space over $B/\mathfrak{n}$. $\square$
Main results (integrally closed ideals)

Let

- \((R, \mathfrak{m})\) be a Gorenstein local ring with \(\dim R = 2\)
- \(I\) an \(\mathfrak{m}\)-primary ideal in \(R\)
- \(I\) contains a parameter ideal \(Q\) s.t. \(I^2 = QI\)
- \(J = Q : I\)

**Proposition 3.3**

Suppose that \(\exists f \in \mathfrak{m}, g \in I, \text{ and } h \in J\) s.t.

\[
IJ = gJ + Ih \quad \text{and} \quad \mathfrak{m}J = fJ + \mathfrak{m}h.
\]

Then \(R(I)\) is an AGG ring.
**Theorem 3.4**

Let \((R, \mathfrak{m})\) be a two-dimensional RLR with \(|R/\mathfrak{m}| = \infty\), and \(I = \bar{I}\). Then \(R(I)\) is an AGG ring.

**Corollary 3.5**

Let \((R, \mathfrak{m})\) be a two-dimensional RLR with \(|R/\mathfrak{m}| = \infty\). Then \(R(\mathfrak{m}^\ell)\) is an AGG ring for \(\forall \ell > 0\).
Thank you for your attention.